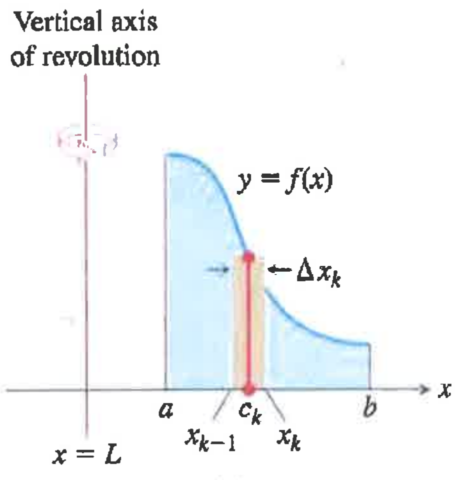
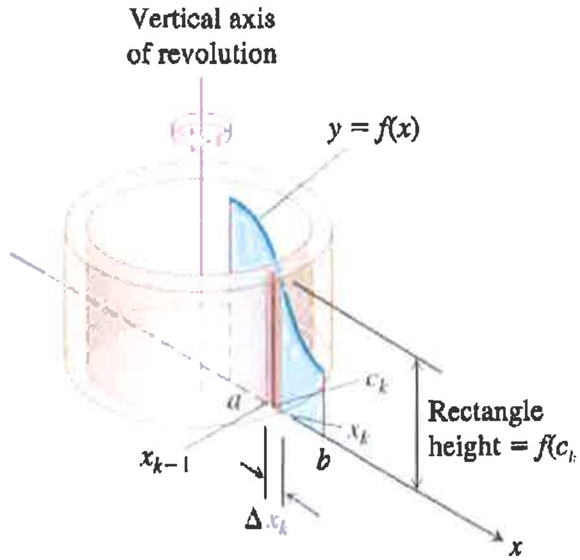
Volumes - Method of Cylindrical Shells





$a = x_0 < x_1 < \cdots x_{k-1} < x_k < \cdots < x_n = b \; , \quad c_k = \dfrac{x_{k-1} + x_k}{2}$

$$V \approx \sum_{k=1}^{n} \left[\pi x_k^2 f(c_k) - \pi x_{k-1}^2 f(c_k) \right] =$$

$$\sum_{k=1}^{n} \pi f(c_k) \left[x_k^2 - x_{k-1}^2 \right] =$$

$$\sum_{k=1}^{n} 2\pi f(c_k) \left[\frac{x_k + x_{k-1}}{2} \right] \left[x_k - x_{k-1} \right] =$$

$$\sum_{k=1}^{n} 2\pi c_k f(c_k) \Delta x_k$$

$$V = \lim_{\|P\| \to 0} \sum_{k=1}^{n} 2\pi c_k f(c_k) \Delta x = \int_a^b 2\pi x f(x)\, dx$$

1. The volume of the solid generated by revolving the region between the x-axis and the graph of a non-negative continuous function $y = f(x)$, $a \leq x \leq b$, about the y-axis is

$$V = \int_a^b 2\pi x\, f(x)\, dx\ .$$

2. The volume of the solid generated by revolving the region between the x-axis and the graph of a non-negative continuous function $y = f(x)$, $a \leq x \leq b$, about the line $x = L$, $L \leq a$ is

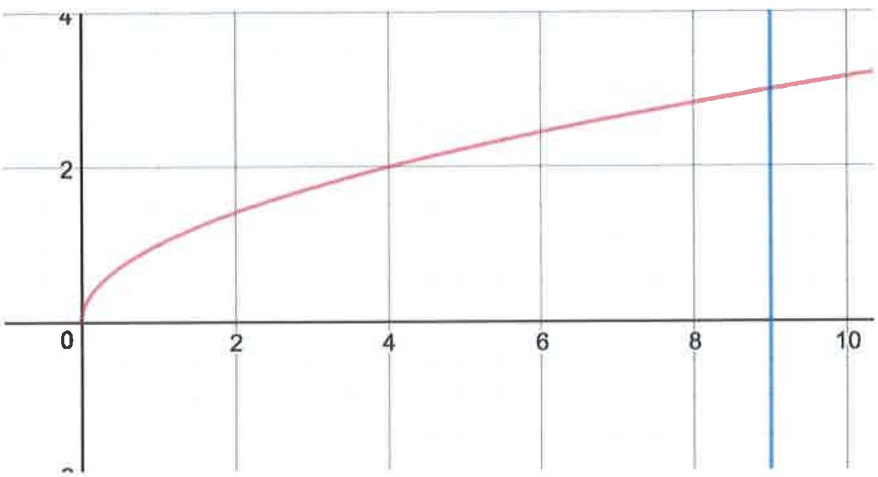
$$V = \int_a^b 2\pi\,(x - L)\, f(x)\, dx\ .$$

3. The volume of the solid generated by revolving the region between the x-axis and the graph of a non-negative continuous function $y = f(x)$, $a \leq x \leq b$, about the line $x = L$, $b \leq L$ is

$$V = \int_a^b 2\pi\,(L - x)\, f(x)\, dx\ .$$

Example 1.

Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$, $y=0$, $0 \leq x \leq 9$ about the y-axis.



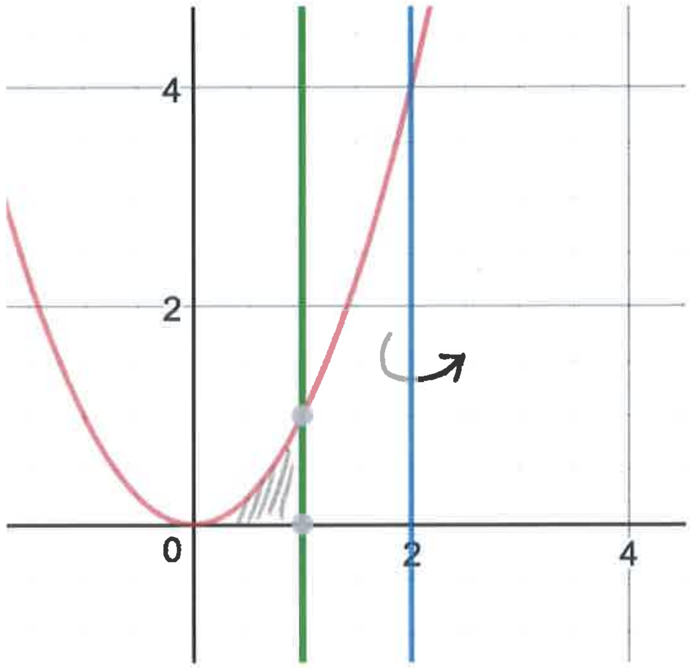
$$V = \int_0^9 2\pi x \sqrt{x} \, dx$$

$$= 2\pi \int_0^9 x^{\frac{3}{2}} \, dx = 2\pi \left. \frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right|_0^9$$

$$= \frac{4}{5} \pi \cdot 3^5 = \frac{572 \pi}{5} .$$

Example 2.

Find the volume of the solid generated by revolving the region bounded by $y = x^2$, $y = 0$, $0 \leq x \leq 1$ about the line $x = 2$.



Use

$$V = \int_a^b 2\pi(L - x) f(x)\,dx$$

$$V = \int_0^1 2\pi(2 - x)\, x^2\, dx$$

$$= \int_0^1 2\pi\left[2x^2 - x^3\right] dx$$

$$= 2\pi\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]\Big|_0^1 = \frac{5\pi}{6}$$